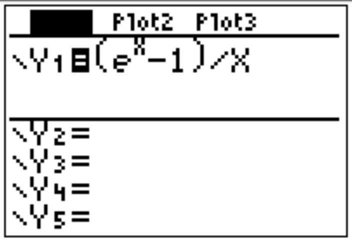
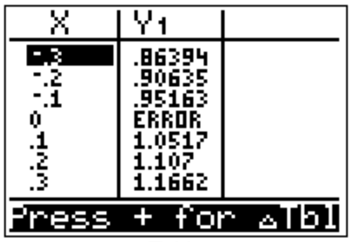
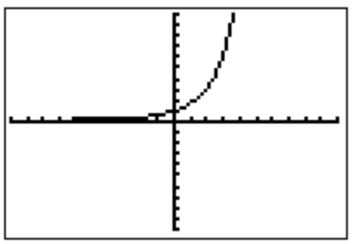


Equation



Table


Graph

Derivative of e^x

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x (1)$$

$$= e^x \ !$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

EXAMPLE 1 **Using the Formula**

Find dy/dx if $y = e^{(x+x^2)}$.

$$\frac{dy}{dx} = e^{(x+x^2)} \cdot (1+2x)$$

Derivative of a^x

$$\underline{a^x = e^{x \ln a}}.$$

$e^{x \ln a} = e^{\ln(a^x)} = a^x$

$e^{x\ln a} = e^{\ln a^x} = a^x$

$$\frac{d}{dx} e^{x \ln a} = \boxed{e^{x \ln a}} \cdot \ln a$$

$$= a^x \ln a$$

$$\frac{d}{dx} e^x = e^x \underbrace{\ln e}_{1}$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

$$y' = 2^t \ln 2$$

$$21 = 2^t \ln 2$$

$$\frac{21}{\ln 2} = 2^t$$

$$\ln\left(\frac{21}{\ln 2}\right) = \ln 2^t$$

$$\ln\left(\frac{21}{\ln 2}\right) = t \ln 2$$

$\ln 21 - \ln 2$

$$\frac{\ln\left(\frac{21}{\ln 2}\right)}{\ln 2} = t \approx 4.921$$

$2^t - 3 = y$

$$\boxed{(4.921, 27.297)}$$

$$\frac{d}{dx}a^x = a^x \ln a$$

EXPLORATION 1 **Leaving Milk on the Counter**

A glass of cold milk from the refrigerator is left on the counter on a warm summer day. Its temperature y (in degrees Fahrenheit) after sitting on the counter t minutes is

$$y = 72 - 30(0.98)^t.$$

Answer the following questions by interpreting y and dy/dt.

1. What is the temperature of the refrigerator? How can you tell? $72-30(.98)^0 = 42°$
2. What is the temperature of the room? How can you tell? $72°$ $\lim_{t\to\infty}(.98)^t = 0$
3. When is the milk warming up the fastest? How can you tell?
   when you take it out of the fridge
4. Determine algebraically when the temperature of the milk reaches 55°F.
5. At what rate is the milk warming when its temperature is 55°F? Answer with an appropriate unit of measure.

$$y' = -30(.98)^t \ln .98$$
$$= -30\ln .98(.98)^t$$
$$\approx .606(.98)^t$$

4. $55 = 72 - 30(.98)^t$

$$-17 = -30(.98)^t$$
$$\frac{17}{30} = (.98)^t$$
$$\ln\left(\frac{17}{30}\right) = t \ln .98$$
$$t = \frac{\ln\left(\frac{17}{30}\right)}{\ln .98} \approx 28 \text{ min}$$

5. $-30\ln(.98)(.98)^t$

$$\approx .343\ {}^\circ\text{F/min}$$

Derivative of ln *x*

$$y = \ln x$$

$$\frac{d}{dx}\left(e^y = x\right)$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

EXAMPLE 3 A Tangent through the Origin

A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

point
$(a, \ln a)$

slope:
$y' = \frac{1}{x}$

$m = \frac{1}{a}$

$= \boxed{\frac{1}{e}}$

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}$$

$$\frac{1}{a} = \frac{\ln a}{a}$$

$$\ln a = 1$$

$$e^{\ln a} = e^1$$

$$a = e$$

Derivative of $\log_a x$

$$\log_a x = \frac{\ln x}{\ln a}.$$

Change of base rule

$$y = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$$

$$y' = \frac{1}{x \ln a}$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

EXAMPLE 4 Going the Long Way with the Chain Rule

Find dy/dx if $y = \log_a a^{\sin x}$.

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a} \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$= \frac{1}{\cancel{a^{\sin x}\ln a}} \cdot \cancel{a^{\sin x}\ln a} \cdot \cos x$$

$$= \cos x$$

Power Rule for Arbitrary Real Powers

$$x^n = e^{n \ln x}.$$

RULE 10 Power Rule for Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

(a) If $y = x^{\sqrt{2}}$,

If $y = (2 + \sin 3x)^{\pi}$

EXAMPLE 6 Finding Domain

If $f(x) = \ln(x - 3)$, find $f'(x)$. State the domain of f'.

EXAMPLE 7 Logarithmic Differentiation

Find dy/dx for $y = x^x$, $x > 0$.

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1+\ln x)$$

$$= x^x(1+\ln x)$$

EXAMPLE 8 How Fast does a Flu Spread?

The spread of a flu in a certain school is modeled by the equation

$$P(t) = \frac{100}{1 + e^{3-t}},$$

where $P(t)$ is the total number of students infected t days after the flu was first noticed. Many of them may already be well again at time t.

(a) Estimate the initial number of students infected with the flu.

(b) How fast is the flu spreading after 3 days?

(c) When will the flu spread at its maximum rate? What is this rate?

a) $P(0) = \frac{100}{1+e^3} \approx 4.743$

≈ 5 students

b) $100\left(1+e^{3-t}\right)^{-1}$

$P'(t) = -100\left(1+e^{3-t}\right)^{-2} \cdot e^{3-t} \cdot -1$

$= \frac{100e^{3-t}}{\left(1+e^{3-t}\right)^2}$

$P'(3) = \frac{100}{4} = 25$ students/day